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WEIGHTED MODEL SPACES AND SCHMIDT SUBSPACES OF HANKEL OPERATORS

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ABSTRACT. For a bounded Hankel matrix Γ, we describe the structure of the Schmidt subspaces of Γ, namely the eigenspaces of $\Gamma^*\Gamma$ corresponding to non zero eigenvalues. We prove that these subspaces are in correspondence with weighted model spaces in the Hardy space on the unit circle. Here we use the term "weighted model space" to describe the range of an isometric multiplier acting on a model space. Further, we obtain similar results for Hankel operators acting in the Hardy space on the real line. Finally, we give a streamlined proof of the Adamyan-Arov-Krein theorem using the language of weighted model spaces.

1. Introduction and main results

1.1. **Overview.** Let $\Gamma = \{\gamma_{j+k}\}_{j,k=0}^{\infty}$ be a Hankel matrix, which we assume to be bounded (but not necessarily compact) on $\ell^2 = \ell^2(\mathbb{Z}_+)$. Under the standard identification between ℓ^2 and the Hardy space $\mathcal{H}^2(\mathbb{T})$ (precise definitions are given below), Γ can be considered as a bounded operator on $\mathcal{H}^2(\mathbb{T})$. It is well known that the kernel of Γ is an invariant subspace of the shift operator, and therefore by Beurling's theorem [3] it is either trivial or can be identified with a subspace of the form $\psi\mathcal{H}^2(\mathbb{T})$ for some inner function ψ.

The question we address in this paper is: *How can one characterise eigenspaces* $\mathrm{Ker}(\Gamma^*\Gamma - s^2 I)$*,* $s > 0$*, as a class of subspaces in the Hardy space?*

Our main result (Theorem 1.5) is that every such eigenspace can be identified with a subspace of the form

$$\mathrm{Ker}(\Gamma^*\Gamma - s^2 I) = p\mathcal{K}_{z\theta} \subset \mathcal{H}^2(\mathbb{T}), \tag{1.1}$$

where θ is an inner function, $\mathcal{K}_{z\theta} = \mathcal{H}^2 \cap (z\theta\mathcal{H}^2)^\perp$ is a model space, and p is an *isometric multiplier* on $\mathcal{K}_{z\theta}$. Furthermore, we show that the action

$$\Gamma : \mathrm{Ker}(\Gamma^*\Gamma - s^2 I) \to \mathrm{Ker}(\Gamma\Gamma^* - s^2 I) \tag{1.2}$$

is given by a simple explicit formula, which is completely determined by s, p and θ. Finally, we prove that the degree of the inner part of p coincides with the total

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multiplicity of the spectrum of $\Gamma^*\Gamma$ in the interval (s^2, ∞). As a simple corollary, we also characterise all eigenspaces of self-adjoint Hankel matrices Γ.

Important precursors to our results are the Adamyan-Arov-Krein (AAK) theory [1] and the description [9] of eigenspaces of $\Gamma^*\Gamma$ for *compact* Hankel operators Γ (in which case the corresponding inner functions θ reduce to finite Blaschke products). However, the isometric multiplier structure (1.1) seems to be a new result even for finite rank Hankel operators.

We were also inspired by works on the structure of *nearly S^*-invariant subspaces* and the kernels of Toeplitz operators; see [13, 14, 19] for early works and [2, 7, 5, 8, 12] for later developments and surveys of the subject. In a companion paper [11] we consider in detail the interesting special case when the spectrum of $\Gamma^*\Gamma$ is finite, solve an inverse spectral problem involving the parameters s, θ and give an explicit description of the corresponding class of symbols.

We shall mention here one important aspect of our proof: it turns out that the natural approach to this problem is to consider Γ together with the "shifted" Hankel matrix $\widetilde{\Gamma} = \{\gamma_{j+k+1}\}_{j,k=0}^{\infty}$. Another feature of our approach is the analysis of a certain new class of subspaces of the Hardy space, which generalise nearly S^*-invariant subspaces.

Let us recall some terminology. A real number $s > 0$ is called a *singular value* of Γ, if s^2 is an eigenvalue of the operator $\Gamma^*\Gamma$. This is slightly wider than the standard definition: we assume neither compactness of Γ nor s^2 to be an isolated point in the spectrum of $\Gamma^*\Gamma$. We note that for a bounded Hankel operator Γ, the operator $\Gamma^*\Gamma|_{(\operatorname{Ker}\Gamma)^\perp}$ can have arbitrary spectrum, see [20].

If s is a singular value of Γ, then a pair $\{\xi, \eta\}$ of elements of ℓ^2 is called a *Schmidt pair* (more precisely, an s-Schmidt pair) of Γ, if it satisfies

$$\Gamma\xi = s\eta, \quad \Gamma^*\eta = s\xi.$$

Clearly, s-Schmidt pairs form a linear subspace of dimension $\dim \operatorname{Ker}(\Gamma^*\Gamma - s^2 I) \leq \infty$. We will call $\operatorname{Ker}(\Gamma^*\Gamma - s^2 I)$ the *Schmidt subspace* of Γ. The problem of description of all s-Schmidt pairs of Γ is equivalent to the problem of the description of the action (1.2).

We briefly describe the structure of this rather long introductory section. In Sections 1.2 and 1.3 we describe the realisation of Hankel matrices on the Hardy space. In Section 1.4 for the purposes of comparison we recall the classical Adamyan-Arov-Krein theorem. In Section 1.5 as a warm-up we consider Hankel operators with inner symbols; this allows us to introduce model spaces into the subject in a natural way. In Section 1.6 we discuss isometric multipliers on model spaces. In Section 1.7 we state and discuss our main result. In Section 1.8 we consider the special case of self-adjoint Hankel matrices Γ and describe their eigenspaces. In Section 1.9 we state the analogue of our main result for Hankel operators acting on the Hardy space $\mathcal{H}^2(\mathbb{R})$ of the real line.

1.2. **Anti-linear operators.** Let us denote by $\mathcal{C}$ the anti-linear operator of complex conjugation on ℓ^2: $\mathcal{C}\xi = \overline{\xi}$. Observe that the Hankel matrix Γ is symmetric and therefore $\Gamma\mathcal{C} = \mathcal{C}\Gamma^*$. It follows that

$$\xi \in \operatorname{Ker}(\Gamma^*\Gamma - s^2 I) \quad \Leftrightarrow \quad \overline{\xi} \in \operatorname{Ker}(\Gamma\Gamma^* - s^2 I)$$

and the *anti-linear* map $\Gamma\mathcal{C}$ maps $\operatorname{Ker}(\Gamma\Gamma^* - s^2 I)$ onto itself. Further, it is clear that the map

$$s^{-1}\Gamma\mathcal{C} : \operatorname{Ker}(\Gamma\Gamma^* - s^2 I) \to \operatorname{Ker}(\Gamma\Gamma^* - s^2 I) \tag{1.3}$$

is an involution.

Thus, the problem of the description of the action (1.2) of Γ is equivalent to the problem of the description of the involution (1.3). As we shall see, the point of view (1.3) offers some advantages. Observe that $(\Gamma\mathcal{C})^2$ is a *linear* operator:

$$(\Gamma\mathcal{C})^2 = \Gamma\mathcal{C}\Gamma\mathcal{C} = \Gamma\Gamma^*.$$

Thus, we can rephrase our aim as follows: we describe the eigenspaces $\operatorname{Ker}((\Gamma\mathcal{C})^2 - s^2 I)$ and the anti-linear involution $s^{-1}\Gamma\mathcal{C}$ on these eigenspaces.

1.3. **Mapping onto the Hardy space.** Let $\mathcal{H}^2(\mathbb{T}) \subset L^2(\mathbb{T})$ be the standard Hardy space, and let $P : L^2(\mathbb{T}) \to \mathcal{H}^2(\mathbb{T})$ be the corresponding orthogonal projection (the Szegő projection). For $f \in \mathcal{H}^2(\mathbb{T})$, let $\widehat{f}_j$ be the j'th Fourier coefficient:

$$\widehat{f}_j = \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} \frac{d\theta}{2\pi}.$$

Conversely, for a sequence $\xi \in \ell^2$, we denote by $\check{\xi} \in \mathcal{H}^2(\mathbb{T})$ the function

$$\check{\xi}(z) = \sum_{j=0}^{\infty} \xi_j z^j, \quad |z| \leq 1.$$

It is standard to study Hankel operators on the Hardy space, i.e. to consider an operator on the Hardy space whose matrix with respect to the standard basis is given by $\{\gamma_{j+k}\}$. In the same way, we map the anti-linear operator $\Gamma\mathcal{C}$ to the Hardy space as follows.

For a *symbol* $u \in \mathcal{H}^2(\mathbb{T})$, we consider the anti-linear Hankel operator H_u, formally defined by

$$H_u f = P(u \cdot \overline{f}), \quad f \in \mathcal{H}^2(\mathbb{T}). \tag{1.4}$$

We are only interested in bounded Hankel operators; it is well known that this corresponds to the symbol u being in the BMOA($\mathbb{T}$) class (see e.g. [18, Theorem 1.2]). We recall that the shift operator S on $\mathcal{H}^2(\mathbb{T})$ is defined as

$$Sf(z) = zf(z), \quad z \in \mathbb{T} \ ,$$

and that the anti-linear Hankel operators H_u are characterised by the identity

$$H_u S = S^* H_u \ .$$

We denote by $\mathbb{1}$ the function in $\mathcal{H}^2(\mathbb{T})$ identically equal to 1; obviously, $u = H_u \mathbb{1}$.

Clearly, the operators $\Gamma\mathcal{C}$ and H_u with $u = \check{\gamma}$ are unitarily equivalent by the Fourier transform:

$$\widecheck{\Gamma\mathcal{C}\xi} = H_u\check{\xi}, \quad u = \check{\gamma}.$$

In particular, H_u^2 is a linear self-adjoint operator on $\mathcal{H}^2(\mathbb{T})$ which is unitarily equivalent to $\Gamma\Gamma^*$. Coming back to the description of the Schmidt pairs for Γ, we can summarise the above discussion as follows:

Proposition 1.1. *A pair $\{\overline{\xi}, \eta\}$ is an s-Schmidt pair of $\Gamma = \{\gamma_{j+k}\}_{j,k=0}^{\infty}$ if and only if*

$$H_u\check{\xi} = s\check{\eta}, \quad \check{\xi} \in \mathrm{Ker}(H_u^2 - s^2 I), \quad u = \check{\gamma}.$$

Thus, our aim is to describe the eigenspaces

$$E_{H_u}(s) := \mathrm{Ker}(H_u^2 - s^2 I) \subset \mathcal{H}^2(\mathbb{T}), \quad s > 0,$$

and the action of H_u on these eigenspaces. For most of the paper, the symbol u is fixed and so we write $E_H(s)$ in place of $E_{H_u}(s)$ when there is no danger of confusion.

In Appendix A, we also discuss the realisation of the linear operator Γ (rather than the anti-linear operator $\Gamma\mathcal{C}$) in the Hardy space.

1.4. **The Adamyan-Arov-Krein theorem.** We recall the classical Adamyan-Arov-Krein (AAK) theorem. Below $\{s_k(\Gamma)\}_{k=1}^{\infty}$ is the ordered sequence of approximation numbers of Γ, i.e.

$$s_k(\Gamma) = \inf\{\|\Gamma - F\| : \mathrm{rank}\, F \leq k\},$$

where the infimum is taken over all linear operators of rank less than or equal to k. Further, $s_\infty(\Gamma) = \lim_{k\to\infty} s_k(\Gamma)$ is the essential norm of Γ. It is well-known that if $s = s_k(\Gamma) > s_\infty(\Gamma)$, then s is a singular value of Γ, i.e. $\mathrm{Ker}(\Gamma^*\Gamma - s^2 I) \neq \{0\}$; furthermore, in this case this space is finite-dimensional.

Similarly, for a bounded anti-linear Hankel operator H_u, we denote

$$s_k(H_u) = \inf\{\|H_u - F\| : \mathrm{rank}\, F \leq k, F \text{ anti-linear}\},$$

where the infimum is taken over all anti-linear operators of rank less than or equal to k, and set $s_\infty(H_u) = \lim_{k\to\infty} s_k(H_u)$.

Adapting the notation of the AAK theorem to our setting (see Proposition 1.1), we quote it as follows.

Theorem 1.2. [1, Theorem 1.2] *Let $s > 0$ be a singular value of a bounded Hankel operator H_u. Then the functions $f \in E_H(s)$, $g = H_u(f)/s$ can be represented as*

$$f(z) = I_+(z)I(z)O(z), \quad g(z) = I_-(z)I(z)O(z), \tag{1.5}$$

where O is an outer function, and I, I_+ and I_- are inner functions. The inner function I is independent of the choice of $f \in E_H(s)$. Further, we have

(i) $\deg I_+ + \deg I_- \leq n-1$, *where* $n = \dim E_H(s)$. *If* n *is finite, then all pairs* (f,g) *can be described by the formula*

$$f(z) = P(z)I(z)\Omega(z), \quad g(z) = z^{n-1}\overline{P(z)}I(z)\Omega(z), \quad |z| = 1, \tag{1.6}$$

where P *is an arbitrary polynomial of degree less than or equal to* $n-1$, *and* Ω *is an outer function which is uniquely defined by* H_u *if we require that* $\Omega(0) > 0$.

(ii) *If* $s = s_k(H_u)$ *and*

$$s_{k-1}(H_u) > s_k(H_u) \geq s_\infty(H_u),$$

then the degree of I *is* k. *Further, in this case there exists a rational function* r *with no poles in* $\overline{\mathbb{D}}$ *such that the rank of* H_r *is* k *and*

$$\|H_u - H_r\| = s.$$

1.5. **Inner symbols and model spaces.** We recall that for an inner function θ on the unit disk, the model space $\mathcal{K}_\theta$ is defined by

$$\mathcal{K}_\theta = \mathcal{H}^2(\mathbb{T}) \cap (\theta\mathcal{H}^2(\mathbb{T}))^\perp.$$

By Beurling's theorem, any proper S^*-invariant subspace of $\mathcal{H}^2(\mathbb{T})$ is a model space.

The basic building blocks of our construction are the Hankel operators H_u whose symbols are inner functions. In this case, as is well known (and easy to prove, see e.g. [18, Theorem 1.2.6]), H_u^2 is an orthogonal projection whose range is a model space. We state this precisely for clarity:

Lemma 1.3. *Let* θ *be an inner function. Then* H_θ^2 *is an orthogonal projection in* $\mathcal{H}^2(\mathbb{T})$, *with* $\operatorname{Ran} H_\theta^2 = \operatorname{Ran} H_\theta = \mathcal{K}_{z\theta}$ *described by*

$$f \in \operatorname{Ran} H_\theta \Leftrightarrow f \in \mathcal{H}^2(\mathbb{T}) \text{ and } \theta\overline{f} \in \mathcal{H}^2(\mathbb{T}).$$

Further, H_θ *acts on* $\operatorname{Ran} H_\theta$ *as an anti-linear involution,*

$$H_\theta f = \theta\overline{f}, \quad f \in \operatorname{Ran} H_\theta.$$

Observe that we have $\theta, \mathbb{1} \in \operatorname{Ran} H_\theta$ for any inner function θ.

To illustrate Lemma 1.3, let us discuss the case of inner functions of finite degree (the degree of θ is defined as $\dim \mathcal{K}_\theta$). An inner function θ has a finite degree k if and only if it is given by a finite Blaschke product

$$\theta(z) = e^{i\alpha}\prod_{j=1}^{k}\frac{z - z_j}{1 - \overline{z_j}z}, \quad |z_j| < 1. \tag{1.7}$$

In this case, the model space $\operatorname{Ran} H_\theta$ can be easily described:

$$\operatorname{Ran} H_\theta = \{P(z)/D(z), \quad P \in \mathbb{C}_k[z]\},$$

where $\mathbb{C}_k[z]$ is the space of all polynomials of degree $\leq k$ and $D(z)$ is the denominator in (1.7), i.e.

$$D(z) = \prod_{j=1}^{k}(1 - \overline{z_j}z).$$

The action of H_θ on $\operatorname{Ran} H_\theta$ in this case is given by

$$H_\theta : P(z)/D(z) \mapsto e^{i\alpha} z^k \overline{P(z)}/D(z), \quad z \in \mathbb{T}. \tag{1.8}$$

1.6. **Isometric multipliers and weighted model spaces.** The following notion will be used throughout the paper. It appeared in the literature in connection with the characterization of kernels of Toeplitz operators, see [14, 13, 19]. Let $F \subset \mathcal{H}^2(\mathbb{T})$ be a closed subspace; a holomorphic function p in the unit disk is called an *isometric multiplier* on F, if for every $f \in F$, the product pf belongs to $\mathcal{H}^2(\mathbb{T})$ and

$$\|pf\| = \|f\|. \tag{1.9}$$

In this case, we write

$$pF := \{pf : f \in F\}.$$

We will mostly use this notion for $F = \mathcal{K}_\theta$, where θ is an inner function. In this case, we will call $p\mathcal{K}_\theta$ a *weighted model space*. (This is not a standard piece of terminology; the adjective *weighted* often implies the choice of a norm different from the standard one, but here in fact the norm is unchanged.)

Observe that if $\theta(0) = 0$, then $\mathbb{1} \in \mathcal{K}_\theta$, and therefore in this case we necessarily have $p \in \mathcal{H}^2(\mathbb{T})$ and $\|p\| = 1$. In [19], Sarason characterized all isometric multipliers of a given model space $\mathcal{K}_\theta$ with $\theta(0) = 0$, proving that these are the functions $p \in \mathcal{H}^2(\mathbb{T})$ of norm one, of the form

$$p(z) = \frac{a(z)}{1 - \theta(z)b(z)}\ , \tag{1.10}$$

where $a, b \in \mathcal{H}^\infty$ are such that $|a|^2 + |b|^2 = 1$ almost everywhere on the unit circle.

Remark 1.4. Let $M = p \operatorname{Ran} H_\theta$, where p is an isometric multiplier on $\operatorname{Ran} H_\theta = \mathcal{K}_{z\theta}$. Then the parameters p and θ in this representation for M are uniquely defined up to unimodular multiplicative constants. This is an easy fact, which follows from Theorem 5.1(ii) below.

1.7. **Main result: the structure of Schmidt subspaces.** For a symbol $a \in L^\infty(\mathbb{T})$, we denote by T_a the Toeplitz operator in $\mathcal{H}^2(\mathbb{T})$, defined by $T_a f = P(af)$. In fact, below we use this notation also for unbounded symbols, but due to the isometricity of the corresponding multipliers, the resulting Toeplitz operators turn out to be bounded on relevant domains.

The following theorem is the main result of the paper. We recall that the total multiplicity of the spectrum of a self-adjoint operator A in an interval Δ is the rank of the spectral projection $\mathcal{E}_A(\Delta)$.

Theorem 1.5. *Let $u \in \mathrm{BMOA}(\mathbb{T})$, let H_u be the anti-linear Hankel operator* (1.4)*, and let $s > 0$ be a singular value of H_u, i.e. $E_H(s) \neq \{0\}$.*

(i) *There exists an inner function θ and an isometric multiplier $p \in \mathcal{H}^2(\mathbb{T})$ on* $\operatorname{Ran} H_\theta$ *such that*

$$E_H(s) = p \operatorname{Ran} H_\theta. \tag{1.11}$$

The functions p and θ can be chosen such that

$$H_u(p) = s\theta p,$$

in which case we have

$$H_u T_p = s T_p H_\theta \quad \text{on } \operatorname{Ran} H_\theta.$$

(ii) *Let φ be the inner factor of p. Then we have*

$$s = \|H_u|_{\varphi \mathcal{H}^2}\|$$

and the degree of φ equals the total multiplicity of the spectrum of the self-adjoint operator $|H_u| = \sqrt{H_u^2}$ in the open interval (s, ∞).

Remark 1.6. (1) Equation $H_u(p) = s\theta p$ is simply a normalisation condition on the unimodular constants in the definition of p, θ (see Remark 1.4).

(2) Part (i) of the theorem shows that the parameters p, θ describe both the structure of the Schmidt subspace $E_H(s)$ and the action of H_u on this subspace. In fact, the action of H_u on $E_H(s)$ is reduced to the simple action of H_θ.

(3) Theorem 1.5(i) extends part (i) of the AAK theorem in two directions. Firstly, Theorem 1.5(i) applies to both finite and infinite dimensional subspaces. Secondly, it provides the isometry of the corresponding multipliers.

(4) The inner function I in (1.5) can be identified with the inner factor φ of p in the representation (1.11). Further, we recognise the action (1.8) in formula (1.6).

(5) For compact H_u, part (i) of the theorem in a less explicit form (and without the isometry property (1.9)) appeared earlier in [9]. Of course, for compact H_u, all Schmidt subspaces are finite dimensional; the action of H_u was expressed in the form (1.8) in [9]. The proof in [9] heavily relied on compactness and on the use of the AAK theory. The proof we give in this paper is both simpler and more general.

(6) If $s = \|H_u\|$, part (ii) of the theorem says that the degree of φ is zero, i.e. p is an outer function. If $|H_u|$ has infinitely many eigenvalues or some essential spectrum in the interval (s, ∞), it says that the degree of φ is infinite.

(7) Our proof of part (i) of the theorem is independent from the AAK theory. It involves the analysis of the Schmidt subspaces of both Γ and $\widetilde{\Gamma}$ and also borrows some elements from the theory of nearly S^*-invariant subspaces. On the other hand, the proof of part (ii) is essentially the adaptation of

the original AAK argument to the language of weighted model spaces, with some simplifications.

Remark 1.7. A natural question is whether any subspace of the form $p\operatorname{Ran}H_\theta$ can appear as a Schmidt subspace of a bounded Hankel operator. Let $p = \varphi p_0$ be the inner-outer factorisation of p; thus, a subspace $p\operatorname{Ran}H_\theta$ is characterised by the triple θ, φ, p_0. In Section 6 we show that any pair of inner functions θ, φ can appear in a description of a Schmidt subspace of a bounded Hankel operator. We don't know whether there are any restrictions on the outer factor p_0, apart from Sarason's condition (1.10).

1.8. **The self-adjoint case.** Here we consider the case of selfadjoint Hankel operators and describe the corresponding eigenspaces. Clearly, a Hankel matrix Γ is self-adjoint if and only if all coefficients γ_j are real. Thus, we consider the Hankel operators H_u with the symbol $u = \check{\gamma}$ having real Fourier coefficients. Then H_u maps the space of functions $\mathcal{H}^2_{\mathrm{real}}$ with *real* Fourier coefficients to itself, and the restriction of H_u onto this space is unitarily equivalent to Γ acting on the ℓ^2 space of real sequences. The following corollary characterizes the eigenspaces of this restriction associated to non-zero eigenvalues.

Corollary 1.8 (the selfadjoint case)**.** *Consider H_u with $u \in \mathrm{BMOA}(\mathbb{T}) \cap \mathcal{H}^2_{\mathrm{real}}$ acting on $\mathcal{H}^2_{\mathrm{real}}$. Let $\lambda \in \mathbb{R}\setminus\{0\}$ be an eigenvalue of H_u. Then there exists an inner function $\theta \in \mathcal{H}^2_{\mathrm{real}}$ and an isometric multiplier $p \in \mathcal{H}^2_{\mathrm{real}}$ on $\operatorname{Ran}H_\theta$ such that*

$$\operatorname{Ker}(H_u - \lambda I) = \{pf : f \in \operatorname{Ran}H_\theta \cap \mathcal{H}^2_{\mathrm{real}}\ ,\ H_\theta f = f\}\ .$$

Under the hypothesis of the corollary, we also have

$$\operatorname{Ker}(H_u + \lambda I) = \{pf : f \in \operatorname{Ran}H_\theta \cap \mathcal{H}^2_{\mathrm{real}}\ ,\ H_\theta f = -f\}.$$

Since

$$\operatorname{Ker}(H_u \pm \lambda I) = p\operatorname{Ker}(H_\theta \pm I),$$

the analysis of the dimension of these eigenspaces reduces to the analysis of the action of H_θ on $\operatorname{Ran}H_\theta$. If $\deg\theta = \infty$, it is clear that both dimensions above are infinite. If $\deg\theta = k < \infty$, then (1.8) shows that the action of H_θ on $\operatorname{Ran}H_\theta$ can be represented by the $(k+1)\times(k+1)$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Thus, we are led to considering the eigenspaces of this matrix corresponding to the eigenvalues ± 1. As in [10], we conclude that

$$|\dim\operatorname{Ker}(H_u - \lambda I) - \dim\operatorname{Ker}(H_u + \lambda I)| \le 1,$$

which recovers the result of [16].

1.9. **Hankel operators on the real line.** Here we briefly discuss the analogous result for Hankel operators acting on the Hardy space $\mathcal{H}^2(\mathbb{R})$ of the real line. Let us first introduce some notation. As usual,

$$\mathcal{H}^2(\mathbb{R}) = \{\mathbf{f} \in L^2(\mathbb{R}) : \widehat{\mathbf{f}}(\xi) = 0,\ \xi < 0\},$$

where $\widehat{\mathbf{f}}$ is the Fourier transform,

$$\widehat{\mathbf{f}}(\xi) = \int_{\mathbb{R}} e^{-i2\pi\xi x}\mathbf{f}(x)dx.$$

As it is standard, we shall consider functions in $\mathcal{H}^2(\mathbb{R})$ as holomorphic functions in the upper half-plane. Let $\mathbf{P}$ be the orthogonal projection from $L^2(\mathbb{R})$ onto $\mathcal{H}^2(\mathbb{R})$. For $\mathbf{u} \in \mathrm{BMOA}(\mathbb{R})$, the Hankel operator $\mathbf{H_u}$ in $\mathcal{H}^2(\mathbb{R})$ is defined by

$$\mathbf{H_u f} = \mathbf{P}(\mathbf{u}\overline{\mathbf{f}}), \quad \mathbf{f} \in \mathcal{H}^2(\mathbb{R}).$$

Since the kernel of $\mathbf{H_u}$ is invariant under the shifts $\mathbf{f}(x) \mapsto e^{ix\xi}\mathbf{f}(x)$, $\xi > 0$, by a theorem of Lax [15] (see also [17, Section 6.5]), similarly to the unit disk case, this kernel can be described as

$$\operatorname{Ker} \mathbf{H_u} = \boldsymbol{\psi}\mathcal{H}^2(\mathbb{R}),$$

where $\boldsymbol{\psi}$ is an inner function in the upper half-plane.

For an inner function $\boldsymbol{\theta}$ in the upper half-plane, we will use the model space $\boldsymbol{\mathcal{K}}_\theta = \operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}} \subset \mathcal{H}^2(\mathbb{R})$, characterised by the condition

$$\mathbf{f} \in \operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}} \text{ if and only if } \boldsymbol{\theta}\overline{\mathbf{f}} \in \mathcal{H}^2(\mathbb{R}).$$

A holomorphic function $\mathbf{p}$ in the upper half plane is called an isometric multiplier on $\operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}}$, if for any $\mathcal{H}^2(\mathbb{R})$, the product $\mathbf{pf}$ belongs to $\mathcal{H}^2(\mathbb{R})$ and

$$\|\mathbf{pf}\| = \|\mathbf{f}\|.$$

In this case, we shall call

$$\mathbf{p} \operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}} = \{\mathbf{pf} : \mathbf{f} \in \operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}}\}$$

the weighted model space, associated with $\mathbf{p}$ and $\boldsymbol{\theta}$; clearly, this is a closed subspace of $\mathcal{H}^2(\mathbb{R})$. We shall denote by $\mathbf{T_p}$ the Toeplitz operator with the symbol $\mathbf{p}$ acting on $\operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}}$, i.e. $\mathbf{T_p f} = \mathbf{pf}$.

In analogy with Theorem 1.5, we have the following result.

Theorem 1.9. *Let* $\mathbf{u} \in \mathrm{BMOA}(\mathbb{R})$*, let* $\mathbf{H_u}$ *be the corresponding anti-linear Hankel operator and let* $s > 0$ *be a singular value of* $\mathbf{H_u}$*, i.e.* $\operatorname{Ker}(\mathbf{H_u}^2 - s^2 I) \neq \{0\}$.

(i) *There exists an inner function* $\boldsymbol{\theta}$ *in the upper half-plane and an isometric multiplier* $\mathbf{p}$ *on* $\operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}}$ *such that*

$$\operatorname{Ker}(\mathbf{H_u}^2 - s^2 I) = \mathbf{p} \operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}}.$$

The functions $\mathbf{p}$ *and* $\boldsymbol{\theta}$ *can be chosen such that*

$$\mathbf{H_u T_p} = s\mathbf{T_p H}_{\boldsymbol{\theta}} \quad \text{on } \operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}}.$$

(ii) *Let* $\boldsymbol{\varphi}$ *be the inner factor of* $\mathbf{p}$. *Then we have*

$$s = \|\mathbf{H_u}|_{\varphi\mathcal{H}^2}\|$$

and the degree of $\boldsymbol{\varphi}$ *equals the total multiplicity of the spectrum of* $|\mathbf{H_u}| = \sqrt{\mathbf{H}_\mathbf{u}^2}$ *in the open interval* (s, ∞).

1.10. **The structure of the paper.** The strategy of the proof is outlined in Section 2, where we introduce the Hardy space anti-linear version K_u of the operator $\widetilde{\Gamma} = \{\gamma_{j+k+1}\}_{j,k=0}^\infty$ and state Theorem 2.2, which is a more detailed version of our main result, involving both H_u and K_u. Further, we discuss a generalisation of nearly S^*-invariant subspaces and state a general geometric result (Theorem 2.3) about such subspaces.

In Section 3 we prove Theorem 2.3. Using this theorem, in Section 4 we prove Theorem 2.2. In Section 5 we prove Corollary 1.8 and Theorem 1.5(i). Section 6 is devoted to a simple example, corresponding to Hankel operators with one or two singular values. In Section 7 we prove Theorem 1.5(ii). In Section 8 by considering a conformal map from the unit disk to the upper half-plane we prove Theorem 1.9. In Appendix A, we reformulate our main result Theorem 1.5(i) in terms of the linear (rather than anti-linear) representation of Hankel operators in the Hardy space.

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2. The strategy of the proof

2.1. **The operator K_u.** Our proof requires another Hankel operator, which is defined by

$$K_u := H_u S = S^* H_u = H_{S^* u}.$$

Clearly, K_u is unitarily equivalent to $\widetilde{\Gamma}\mathcal{C}$, where $\widetilde{\Gamma} = \{\widehat{u}_{j+k+1}\}_{j,k=0}^\infty$. We have a crucial identity

$$K_u^2 = H_u S S^* H_u = H_u^2 - (\cdot, u)u \ , \tag{2.1}$$

where $(\cdot, u)u$ denotes the rank one operator corresponding to the element u. For $s > 0$, similarly to $E_{H_u}(s)$, we denote

$$E_{K_u}(s) = \operatorname{Ker}(K_u^2 - s^2 I).$$

We shall write $E_K(s)$ instead of $E_{K_u}(s)$ when the choice of u is clear from the context. We start with the basic statement which shows that (as a consequence of (2.1)) the eigenspaces $E_H(s)$ and $E_K(s)$ differ by the one-dimensional subspace spanned by u.

Lemma 2.1. *Let $s > 0$ be a singular value of either H_u or K_u, i.e.*

$$E_H(s) + E_K(s) \neq \{0\}.$$

Then one (and only one) of the following properties holds:

(1) *$u \not\perp E_H(s)$, and $E_K(s) = E_H(s) \cap u^\perp$;*
(2) *$u \not\perp E_K(s)$, and $E_H(s) = E_K(s) \cap u^\perp$.*

In case (1) above, we will say that the singular value s is *H-dominant*; in case (2) we will say that s is *K-dominant.*

This lemma was established in [9] in the case when H_u and K_u are compact, but in fact the proof does not use compactness. In any case, in Section 4 for completeness we repeat the proof.

We note that for a general pair of operators H_u^2, K_u^2, satisfying the rank-one identity (2.1), three scenarios are possible: (1), (2) of Lemma 2.1 and

(3) $u \perp E_H(s)$, $u \perp E_K(s)$, and $E_H(s) = E_K(s)$.

Identity $K_u = S^* H_u$ ensures that (3) is not possible, see the proof of Lemma 2.1.

2.2. **The Schmidt subspaces of H_u and K_u.** If $g \in \mathcal{H}^2(\mathbb{T})$, $g \neq 0$, and if $g/\|g\|$ is an isometric multiplier of $\operatorname{Ran} H_\theta$, below we write for simplicity $g \operatorname{Ran} H_\theta$ instead of $(g/\|g\|) \operatorname{Ran} H_\theta$.

The following theorem describes the Schmidt subspaces of both H_u and K_u. The two cases below correspond to the two cases in Lemma 2.1.

Theorem 2.2. *Let $u \in \mathrm{BMOA}(\mathbb{T})$ and let the anti-linear Hankel operators H_u, K_u be as defined above. Let $s > 0$ be a singular value of either H_u or K_u.*

(i) *Let s be H-dominant (i.e. $u \not\perp E_H(s)$); denote by $\mathbb{1}_s$ (resp. by u_s) the orthogonal projection of $\mathbb{1}$ (resp. of u) onto $E_H(s)$. Then $u_s = s\psi_s \mathbb{1}_s$ for an inner function ψ_s. The function $\mathbb{1}_s/\|\mathbb{1}_s\|$ is an isometric multiplier on $\operatorname{Ran} H_{\psi_s}$ and the subspaces $E_H(s)$, $E_K(s)$ are given by*

$$\begin{aligned} E_H(s) &= \mathbb{1}_s \operatorname{Ran} H_{\psi_s}, \\ E_K(s) &= \mathbb{1}_s(\operatorname{Ran} H_{\psi_s} \cap \psi_s^\perp) = \mathbb{1}_s \operatorname{Ran} K_{\psi_s}. \end{aligned} \tag{2.2}$$

The action of H_u and K_u on these subspaces is given by

$$H_u T_{\mathbb{1}_s} = s T_{\mathbb{1}_s} H_{\psi_s} \quad \text{on } \operatorname{Ran} H_{\psi_s}, \tag{2.3}$$

$$K_u T_{\mathbb{1}_s} = s T_{\mathbb{1}_s} K_{\psi_s} \quad \text{on } \operatorname{Ran} K_{\psi_s} = \operatorname{Ran} H_{\psi_s} \cap \psi_s^\perp. \tag{2.4}$$

The inner function ψ_s is uniquely defined by (2.2) and (2.3).

(ii) *Let s be K-dominant (i.e. $u \not\perp E_K(s)$); denote by $\widetilde{u}_s$ the orthogonal projection of u onto $E_K(s)$. Then $K_u(\widetilde{u}_s) = s\widetilde{\psi}_s \widetilde{u}_s$ for an inner function $\widetilde{\psi}_s$. The function $\widetilde{u}_s/\|\widetilde{u}_s\|$ is an isometric multiplier on $\operatorname{Ran} H_{\widetilde{\psi}_s}$. The subspaces $E_H(s)$, $E_K(s)$*

are given by

$$E_H(s) = \widetilde{u}_s S(\operatorname{Ran} K_{\widetilde{\psi}_s}) = \widetilde{u}_s(\operatorname{Ran} H_{\widetilde{\psi}_s} \cap \mathbb{1}^\perp),$$
$$E_K(s) = \widetilde{u}_s \operatorname{Ran} H_{\widetilde{\psi}_s}. \tag{2.5}$$

The action of H_u and K_u on these subspaces is given by

$$H_u T_{\widetilde{u}_s} = s T_{\widetilde{u}_s} S H_{\widetilde{\psi}_s} \quad \text{on } S(\operatorname{Ran} K_{\widetilde{\psi}_s}) = \operatorname{Ran} H_{\widetilde{\psi}_s} \cap \mathbb{1}^\perp, \tag{2.6}$$
$$K_u T_{\widetilde{u}_s} = s T_{\widetilde{u}_s} H_{\widetilde{\psi}_s} \quad \text{on } \operatorname{Ran} H_{\widetilde{\psi}_s}. \tag{2.7}$$

The inner function $\widetilde{\psi}_s$ is uniquely defined by (2.5) *and* (2.7)*.*

Let us briefly explain how Theorem 1.5 follows from Theorem 2.2 (we give the full proof in Section 5). In the H-dominant case, the first part of Theorem 2.2 immediately provides the decomposition (1.11) with $p = \frac{\mathbb{1}_s}{\|\mathbb{1}_s\|}$ and $\theta = \psi_s$. In the K-dominant case, the second part of Theorem 2.2 provides $E_H(s) = \widetilde{u}_s S(\operatorname{Ran} K_{\widetilde{\psi}_s})$. It is not difficult to derive the desired decomposition (1.11) from here. Uniqueness is an easy consequence of Remark 1.4.

2.3. **A generalisation of nearly S^*-invariant subspaces.** The proof of Theorem 2.2 is a consequence of the following general statement.

Theorem 2.3. *Let M be a closed subspace of $\mathcal{H}^2(\mathbb{T})$, and let $p, q \in M$ be unit vectors such that*

$$M \cap p^\perp = S(M \cap q^\perp). \tag{2.8}$$

(i) *Then $q = \theta p$ with an inner function θ.*
(ii) *Assume in addition that for any $f \in \operatorname{Ran} H_\theta \cap \mathcal{H}^\infty(\mathbb{T})$ we have the implication*

$$pf \in M, \quad f \perp \mathbb{1} \quad \Rightarrow \quad pf \perp p. \tag{2.9}$$

Then p is an isometric multiplier on $\operatorname{Ran} H_\theta$ and

$$M = p \operatorname{Ran} H_\theta\ . \tag{2.10}$$

Both the statement and the proof of Theorem 2.3 are closely related to the theory of the *nearly S^*-invariant subspaces*, see [14, 19]. These are the subspaces M satisfying $f \in M \cap \mathbb{1}^\perp \Rightarrow S^* f \in M$. By [14, Proposition 3], every nearly S^*-invariant subspace M is either a weighted model space or a subspace of the form $M = \psi \mathcal{H}^2(\mathbb{T})$ with inner ψ.

However, a weighted model space is not necessarily nearly S^*-invariant. For example, if $p(0) = 0$, we have $M \perp \mathbb{1}$, and so nearly S^*-invariance of M would mean $S^* M \subset M$, which is only possible if $M = \{0\}$. In any case, M is S^*-invariant on a subspace of codimension 1, and so it shares many properties with nearly S^*-invariant subspaces.

To prove Theorem 2.2(i), we take

$$M = E_H(s), \quad p = \mathbb{1}_s / \|\mathbb{1}_s\|, \quad q = u_s / \|u_s\|.$$

To prove Theorem 2.2(ii), we take

$$M = E_K(s), \quad p = \widetilde{u}_s/\|\widetilde{u}_s\|, \quad q = K_u\widetilde{u}_s/\|K_u\widetilde{u}_s\|.$$

The crucial geometric property (2.8) in both cases is established in Lemma 4.1 below.

2.4. **Some preliminary identities.** Throughout the rest of the paper, we use the following identity:

$$H_u(fg) = P(\overline{f}H_u g), \quad f \in \mathcal{H}^\infty(\mathbb{T}), \quad g \in \mathcal{H}^2(\mathbb{T}). \tag{2.11}$$

The proof is a direct calculation:

$$H_u(fg) = P(u\overline{f}\overline{g}) = P(\overline{f}u\overline{g}) = P(\overline{f}P(u\overline{g})) = P(\overline{f}H_u g).$$

Of course, this identity can also be applied with K_u in place of H_u. Identity (2.11) can be alternatively written in terms of the Toeplitz operators T_f, $T_{\overline{f}}$ as

$$H_u T_f = T_{\overline{f}} H_u. \tag{2.12}$$

We also use the fact that both H_u and K_u satisfy the following symmetry condition:

$$(H_u f, g) = (H_u g, f), \quad f, g \in \mathcal{H}^2(\mathbb{T}),$$

which follows directly from the definition of H_u.

3. Proof of Theorem 2.3

3.1. **Factorisation** $q = \theta p$**.** 1) For every $\zeta \in \mathbb{D}$, denote by f_ζ the reproducing kernel of M, namely $f_\zeta \in M$ and

$$\forall h \in M, \quad h(\zeta) = (h, f_\zeta)\ .$$

Denote by g_ζ the orthogonal projection of f_ζ onto $M \cap q^\perp$, and by h_ζ the orthogonal projection of f_ζ onto $M \cap p^\perp$. By assumption, we have $h_\zeta = SS^*h_\zeta$, with $S^*h_\zeta \in M \cap q^\perp$, and $Sg_\zeta \in M \cap p^\perp$, hence

$$h_\zeta(\zeta) = \zeta S^*h_\zeta(\zeta) = \zeta(S^*h_\zeta, f_\zeta) = \zeta(S^*h_\zeta, g_\zeta) = \zeta(h_\zeta, Sg_\zeta) = \zeta(f_\zeta, Sg_\zeta) = |\zeta|^2\overline{g_\zeta(\zeta)}\ .$$

Since f_ζ is the reproducing kernel of M, we have

$$\|g_\zeta\|^2 = (g_\zeta, f_\zeta) = g_\zeta(\zeta)\ ,\ \|h_\zeta\|^2 = (h_\zeta, f_\zeta) = h_\zeta(\zeta),$$

and so we conclude that

$$\|h_\zeta\|^2 = |\zeta|^2\|g_\zeta\|^2 \le \|g_\zeta\|^2. \tag{3.1}$$

2) Let us first prove that $q = \theta p$ with some $\theta \in \mathcal{H}^\infty$, $\|\theta\|_{\mathcal{H}^\infty} \le 1$. Since q and p are non zero holomorphic functions on the unit disk $\mathbb{D}$, it suffices to prove that

$$\forall \zeta \in \mathbb{D}\ ,\ |q(\zeta)| \le |p(\zeta)|\ .$$

We shall obtain this from (3.1) by a duality argument.

Expanding f_ζ with respect to two orthogonal decompositions

$$M = (M \cap q^\perp) \oplus \mathrm{span}\{q\}\ ;\ M = (M \cap p^\perp) \oplus \mathrm{span}\{p\}\ ,$$

we obtain

$$f_\zeta = g_\zeta + \lambda q = h_\zeta + \mu p \ . \tag{3.2}$$

The constants λ and μ satisfy

$$\|f_\zeta\|^2 = \|g_\zeta\|^2 + |\lambda|^2 = \|h_\zeta\|^2 + |\mu|^2.$$

Using (3.1), we conclude that $|\mu| \geq |\lambda|$. Taking an inner product of (3.2) with q and with p, we get

$$q(\zeta) = (q, f_\zeta) = (q, \lambda q) = \overline{\lambda},$$
$$p(\zeta) = (p, f_\zeta) = (p, \mu p) = \overline{\mu}.$$

It follows that $|q(\zeta)| \leq |p(\zeta)|$, as required.

3) Finally, let us check that θ is inner. Using $q = \theta p$, we see that

$$0 = \|p\|^2 - \|q\|^2 = \int_{-\pi}^{\pi} (1 - |\theta(e^{it})|^2)|p(e^{it})|^2 \frac{dt}{2\pi}.$$

As $|\theta(e^{it})|^2 \leq 1$, we see that the integrand above vanishes for a.e. t. Since $p(e^{it}) \neq 0$ for a.e. t, we conclude that $|\theta(e^{it})|^2 = 1$ for a.e. t.

3.2. p **is an isometric multiplier on** $\operatorname{Ran} H_\theta$**.** 1) Denote

$$F = \operatorname{span}\{(S^*)^n \theta : n \geq 0\},$$

where span denotes the set of all finite linear combinations. It is easy to see that

$$\operatorname{Clos} F = \operatorname{Ran} H_\theta.$$

Let us prove that

$$pF \subset M.$$

Let $f \in F$; assume that $pf \in M$ and consider the element $p(f - f(0)) \in M$. Since $f - f(0) \perp \mathbb{1}$, by the geometric assumption (2.9) we have $p(f - f(0)) \perp p$. From (2.8) it follows that $p(f - f(0)) = zg$ with some $g \in M \cap q^\perp$ and so $pS^* f = g \in M$. Thus, we have an implication

$$f \in F, \quad pf \in M \quad \Rightarrow \quad pS^* f \in M.$$

Since $q = p\theta \in M$, we can apply this implication to $f = \theta$, then to $f = S^*\theta$, etc, to obtain $p(S^*)^n\theta \in M$ for all $n \geq 0$.

2) For $f \in F$, write

$$pf = pf(0) + p(f - f(0)).$$

By the orthogonality assumption (2.9), the two terms in the right hand side are orthogonal to one another, and so

$$\|pf\|^2 = |f(0)|^2 + \|p(f - f(0))\|^2 = |f(0)|^2 + \|pS^* f\|^2. \tag{3.3}$$

On the other hand, we obviously have

$$\|f\|^2 = |f(0)|^2 + \|S^* f\|^2;$$

subtracting, we obtain

$$\|pf\|^2 - \|f\|^2 = \|pS^*f\|^2 - \|S^*f\|^2.$$

Thus, we have an implication

$$f \in F, \quad \|pf\| = \|f\| \quad \Rightarrow \quad \|pS^*f\| = \|S^*f\|.$$

Applying this to $f = \theta$, then to $f = S^*\theta$, etc., we obtain

$$\|p(S^*)^n\theta\| = \|(S^*)^n\theta\|, \quad \forall n \geq 0. \tag{3.4}$$

3) In order to extend the above relation to linear combinations of elements $(S^*)^n\theta$, we need to prove that the set

$$F_0 = \{f \in F : \|pf\| = \|f\|\}$$

is linear. Here we use the argument of [14]. As a first step, let us check the inequality

$$\|pf\| \geq \|f\| \quad \forall f \in F. \tag{3.5}$$

Rewrite (3.3) as

$$\|pf\|^2 = |\widehat{f}_0|^2 + \|pS^*f\|^2$$

and apply this to S^*f in place of f:

$$\|pS^*f\|^2 = |\widehat{f}_1|^2 + \|p(S^*)^2f\|^2.$$

Iterating and summing, we obtain

$$\|pf\|^2 = \sum_{n=0}^{N} |\widehat{f}_n|^2 + \|p(S^*)^{N+1}f\|^2 \geq \sum_{n=0}^{N} |\widehat{f}_n|^2.$$

Sending $N \to \infty$, we obtain (3.5).

4) Consider the linear operator

$$T_{1/p} : pF \to F.$$

By (3.5), this is a contraction. It is straightforward to see that F_0 can be characterised as the image of $\mathrm{Ker}(T_{1/p}^*T_{1/p} - I)$ under $T_{1/p}$; thus, F_0 is linear. Thus, the isometry relation (3.4) extends to all linear combinations of elements $(S^*)^n\theta$. In other words, we obtain that the map

$$T_p : F \to pF \subset M$$

is an isometry. Since F is dense in $\mathrm{Ran}\, H_\theta$, this map extends as an isometry

$$T_p : \mathrm{Ran}\, H_\theta \to p\,\mathrm{Ran}\, H_\theta \subset M.$$

3.3. **Proof of $p\,\mathrm{Ran}\,H_\theta = M$.** Consider the subspace

$$V = M \cap (p\,\mathrm{Ran}\,H_\theta)^\perp.$$

Our aim is to prove that $V = \{0\}$.

1) Let us first prove that V is S^*-invariant. Let $h \in V$. Then $h \in M \cap p^\perp$, and so $S^*h \in M \cap q^\perp$. It suffices to check that $(S^*h, h') = 0$ for all $h' = pf$, $f \in \mathrm{Ran}\,H_\theta \cap \mathcal{H}^\infty$. For such f, write $\overline{f}\theta = c\mathbb{1} + zw$ with $w \in \mathrm{Ran}\,H_\theta \cap \mathcal{H}^\infty$; then

$$f = \overline{c}\theta + \overline{zw}\theta,$$

and so

$$h' = \overline{c}p\theta + p\overline{zw}\theta = \overline{c}q + p\overline{zw}\theta.$$

Now

$$(S^*h, h') = c(S^*h, q) + (S^*h, p\overline{zw}\theta).$$

Here the first term vanishes as $S^*h \in M \cap q^\perp$. For the second term, we have $SS^*h = h$ and so

$$(S^*h, p\overline{zw}\theta) = (h, p\overline{w}\theta) = 0,$$

because $h \perp p\,\mathrm{Ran}\,H_\theta$.

2) Thus, V is S^*-invariant and $V \perp \mathbb{1}$ (as $V \subset M \cap p^\perp \subset S(M \cap q^\perp)$). But any subspace satisfying these two conditions is trivial, since for any $h \in V$ we have

$$\widehat{h}(n) = (h, S^n\mathbb{1}) = ((S^*)^n h, \mathbb{1}) = 0$$

for all n. Thus, $V = \{0\}$. The proof of Theorem 2.3 is complete.

4. Proof of Theorem 2.2

4.1. **Lemmas on subspaces.**

Proof of Lemma 2.1. 1) Assume $u \not\perp E_H(s)$; let us prove that $u \perp E_K(s)$ and

$$E_K(s) = E_H(s) \cap u^\perp.$$

Because $K_u^2 = H_u^2 - (\cdot, u)u$, it is clear that

$$E_H(s) \cap u^\perp = E_K(s) \cap u^\perp.$$

In particular, $E_H(s) \cap u^\perp \subset E_K(s)$; let us prove the converse inclusion. Let $h \in E_K(s)$, and let $h' \in E_H(s)$ be such that $(h', u) \neq 0$. Then

$$\begin{aligned} s^2(h, h') &= (K_u^2 h, h') = (H_u^2 h, h') - (h, u)(u, h') \\ &= (h, H_u^2 h') - (h, u)(u, h') = s^2(h, h') - (h, u)(u, h')\ , \end{aligned}$$

which implies that h is orthogonal to u. Hence $E_K(s) \perp u$, and consequently

$$E_K(s) = E_K(s) \cap u^\perp = E_H(s) \cap u^\perp \subset E_H(s),$$

as required.

2) Assume $u \not\perp E_K(s)$; let us prove that $u \perp E_H(s)$ and

$$E_H(s) = E_K(s) \cap u^\perp.$$

By the same reasoning as above we have $E_K(s) \cap u^\perp \subset E_H(s)$. To prove the converse inclusion, let $h \in E_H(s)$ and let $h' \in E_K(s)$ with $(h', u) \neq 0$. Then, as at the previous step, we have

$$s^2(h, h') = (H_u^2 h, h') = (K_u^2 u, h') + (h, u)(u, h') = s^2(h, h') + (h, u)(u, h'),$$

and so $h \perp u$. This gives the inclusion $E_H(s) \subset E_K(s) \cap u^\perp$.

3) Finally, let us prove that u cannot be orthogonal to both $E_H(s)$ and $E_K(s)$. If it is, then

$$E_H(s) = E_H(s) \cap u^\perp = E_K(s) \cap u^\perp = E_K(s).$$

Denote this subspace by V; let us prove that $V = \{0\}$. Both H_u and K_u are anti-linear isomorphisms on V. Given $h \in V$, write $h = H_u h'$ with $h' \in V$. Then $S^* h = K_u h' \in V$. Furthermore,

$$0 = (h', u) = (h', H_u \mathbb{1}) = (\mathbb{1}, H_u h') = (\mathbb{1}, h) \ .$$

Hence $S^*(V) \subset V$ and $V \perp \mathbb{1}$. Thus, as at the last stage of the proof of Theorem 2.3 above, we obtain $V = \{0\}$. □

The following lemma is fundamental for our construction. It will allow us to check the crucial "geometric" hypothesis $M \cap p^\perp = S(M \cap q^\perp)$ of Theorem 2.3.

Lemma 4.1. *Let $E_H(s) + E_K(s) \neq \{0\}$. Then*

$$S(E_H(s) \cap u^\perp) = E_H(s) \cap \mathbb{1}^\perp, \qquad \text{if } s \text{ is } H\text{-dominant},$$
$$S(E_K(s) \cap (K_u u)^\perp) = E_K(s) \cap u^\perp, \qquad \text{if } s \text{ is } K\text{-dominant}.$$

Proof. 1) Let s be H-dominant; let us check that $SE_K(s) \subset E_H(s) \cap \mathbb{1}^\perp$. Let $g \in E_K(s)$; we need to prove that $H_u^2 Sg = s^2 Sg$. In order to do this, observe that

$$S^*(H_u^2 Sg - s^2 Sg) = S^* H_u^2 Sg - s^2 S^* Sg = K_u^2 g - s^2 g = 0,$$

and so $H_u^2 Sg - s^2 Sg \in \operatorname{Ker} S^*$, i.e. $H_u^2 Sg - s^2 Sg = c\mathbb{1}$. Let us compute c:

$$c = (H_u^2 Sg - s^2 Sg, \mathbb{1}) = (H_u^2 Sg, \mathbb{1}) = (H_u \mathbb{1}, H_u Sg) = (u, K_u g) = 0,$$

because $K_u g \in E_K(s)$. We have proved that $Sg \in E_H(s)$. Finally, it is obvious that $SE_K(s) \perp \mathbb{1}$.

2) Let s be H-dominant; let us check that $E_H(s) \cap \mathbb{1}^\perp \subset SE_K(s)$. Let $h \in E_H(s) \cap \mathbb{1}^\perp$; then $h = Sg$ and $g = S^* h$. We need to check that $g \in E_K(s)$, i.e. that $K_u^2 g = s^2 g$. We have

$$K_u^2 g = S^* H_u^2 Sg = S^* H_u^2 h = s^2 S^* h = s^2 g,$$

as required.

3) Let s be K-dominant; let us check that $S(E_K(s) \cap (K_u u)^\perp) \subset E_H(s)$. Let $h \in E_K(s) \cap (K_u u)^\perp$ and $g = Sh$. In order to prove that $H_u^2 g = s^2 g$, consider

$$S^*(H_u^2 g - s^2 g) = S^* H_u^2 Sh - s^2 S^* g = K_u^2 h - s^2 h = 0.$$

It follows that $H_u^2 g - s^2 g = c\mathbb{1}$. Let us compute c:

$$c = (H_u^2 g - s^2 g, \mathbb{1}) = (H_u^2 g, \mathbb{1}) = (H_u \mathbb{1}, H_u g)$$
$$= (u, H_u S h) = (u, K_u h) = (h, K_u u) = 0,$$

as required.

4) Let s be K-dominant; let us check that $E_H(s) \subset S(E_K(s) \cap (K_u u)^\perp)$. Take $g \in E_H(s)$; let us first check that $(g, \mathbb{1}) = 0$. Since $H_u g \in E_H(s) \subset u^\perp$, we have

$$0 = (u, H_u g) = (H_u \mathbb{1}, H_u g) = (H_u^2 g, \mathbb{1}) = s^2 (g, \mathbb{1}),$$

as claimed. Thus, $g = Sh$; let us prove that $h \in E_K(s)$ and $h \perp K_u u$. To see the first inclusion, we compute

$$K_u^2 h - s^2 h = S^* H_u^2 S h - s^2 h = S^* H_u^2 g - s^2 h = s^2 S^* g - s^2 h = 0.$$

Finally,

$$(h, K_u u) = (u, K_u h) = (u, H_u S h) = (u, H_u g) = 0.$$

□

4.2. **Proof of Theorem 2.2(i).** 1) Let P_s be the orthogonal projection onto $E_H(s)$. Then P_s commutes with H_u and therefore

$$u_s = P_s u = P_s H_u \mathbb{1} = H_u P_s \mathbb{1} = H_u \mathbb{1}_s.$$

Furthermore, we have

$$\|u_s\|^2 = (H_u \mathbb{1}_s, H_u \mathbb{1}_s) = (H_u^2 \mathbb{1}_s, \mathbb{1}_s) = s^2 \|\mathbb{1}_s\|^2.$$

Next, let us apply Theorem 2.3(i) with $M = E_H(s)$, $p = \mathbb{1}_s/\|\mathbb{1}_s\|$ and $q = u_s/\|u_s\|$. The geometric hypothesis $M \cap p^\perp = S(M \cap q^\perp)$ is satisfied by Lemma 4.1.

Thus, we obtain

$$\frac{u_s}{\|u_s\|} = \psi_s \frac{\mathbb{1}_s}{\|\mathbb{1}_s\|}$$

with some inner function ψ_s. This can be rewritten in the form

$$u_s = s \psi_s \mathbb{1}_s.$$

2) Let us apply Theorem 2.3(ii) with the same parameters. The orthogonality assumption (2.9) is evidently satisfied:

$$(pf, p) = (pf, \mathbb{1}_s)/\|\mathbb{1}_s\| = (pf, \mathbb{1})/\|\mathbb{1}_s\| = 0$$

if $f(0) = 0$. This yields the description $E_H(s) = \mathbb{1}_s \operatorname{Ran} H_{\psi_s}$ of our Schmidt subspace and the formula $\|f \mathbb{1}_s\| = \|f\| \|\mathbb{1}_s\|$ for all $f \in \operatorname{Ran} H_{\psi_s}$.

Let us check formula (2.3) for the action of H_u on $E_H(s)$ It suffices to check it on the dense set of elements $f \in \operatorname{Ran} H_{\psi_s} \cap \mathcal{H}^\infty$. We have, using (2.11)

$$H_u T_{\mathbb{1}_s} f = H_u (f \mathbb{1}_s) = P(\overline{f} H_u \mathbb{1}_s) = P(\overline{f} u_s) = s P(\overline{f} \psi_s \mathbb{1}_s).$$

Since $\overline{f} \psi_s \in \mathcal{H}^\infty(\mathbb{T})$, we obtain (2.3):

$$H_u T_{\mathbb{1}_s} f = s \overline{f} \psi_s \mathbb{1}_s = s \mathbb{1}_s H_{\psi_s} f = s T_{\mathbb{1}_s} H_{\psi_s} f.$$

3) It remains to describe the subspace $E_K(s)$ and the action of K_u on this subspace. By Lemma 2.1, we have $E_K(s) = E_H(s) \cap u^\perp$; on the other hand, by Theorem 2.3, multiplication by $\mathbb{1}_s/\|\mathbb{1}_s\|$ is a unitary operator from $\operatorname{Ran} H_{\psi_s}$ to $E_H(s)$. Thus, for $f \in \operatorname{Ran} H_{\psi_s}$, the product $\mathbb{1}_s f$ belongs to $E_K(s)$ if and only if

$$0 = (\mathbb{1}_s f, u) = (\mathbb{1}_s f, u_s) = s(\mathbb{1}_s f, \mathbb{1}_s \psi_s) = s\|\mathbb{1}_s\|^2 (f, \psi_s),$$

i.e. if and only if $f \in \operatorname{Ran} H_{\psi_s} \cap \psi_s^\perp$. But $f \perp \psi_s$ can be equivalently rewritten as $\overline{f}\psi_s \perp \mathbb{1}$; this relation means that $f = H_{\psi_s}(zw)$ with some $w \in \operatorname{Ran} H_{\psi_s}$. Thus,

$$\operatorname{Ran} H_{\psi_s} \cap \psi_s^\perp = \operatorname{Ran} H_{\psi_s} S = S^* \operatorname{Ran} H_{\psi_s} = \operatorname{Ran} K_{\psi_s},$$

as claimed.

Let us check formula (2.4) for the action of K_u on $E_K(s)$. Let $f \in \operatorname{Ran} H_{\psi_s} \cap \psi_s^\perp$; then $H_{\psi_s} f \perp \mathbb{1}$, and so

$$S^*(\mathbb{1}_s H_{\psi_s} f) = \mathbb{1}_s S^* H_{\psi_s} f = \mathbb{1}_s K_{\psi_s} f.$$

Now using formula (2.3) for the action of H_u on $E_H(s)$ it follows that

$$K_u(\mathbb{1}_s f) = S^* H_u(\mathbb{1}_s f) = s S^*(\mathbb{1}_s H_{\psi_s} f) = s \mathbb{1}_s K_{\psi_s} f,$$

as required.

4.3. **Proof of Theorem 2.2(ii).** 1) Let P_s be the orthogonal projection onto $E_K(s)$. Then P_s commutes with K_u and therefore $P_s K_u u = K_u P_s u = K_u \widetilde{u}_s$. Furthermore, we have

$$\|K_u \widetilde{u}_s\|^2 = (K_u^2 \widetilde{u}_s, \widetilde{u}_s) = s^2 \|\widetilde{u}_s\|^2.$$

Let us apply Theorem 2.3(i) with $M = E_K(s)$, $p = \widetilde{u}_s/\|\widetilde{u}_s\|$ and $q = K_u\widetilde{u}_s/\|K_u\widetilde{u}_s\|$. The geometric condition $M \cap p^\perp = S(M \cap q^\perp)$ is satisfied by Lemma 4.1. We obtain

$$\frac{K_u \widetilde{u}_s}{\|K_u \widetilde{u}_s\|} = \widetilde{\psi}_s \frac{\widetilde{u}_s}{\|\widetilde{u}_s\|}$$

with some inner function $\widetilde{\psi}_s$. This can be rewritten as

$$K_u \widetilde{u}_s = s \widetilde{\psi}_s \widetilde{u}_s.$$

2) We would like to apply Theorem 2.3(ii) with the same parameters. We need to check the orthogonality assumption (2.9): for $f \in \operatorname{Ran} H_{\widetilde{\psi}_s} \cap \mathcal{H}^\infty$, $\widetilde{u}_s f \in E_K(s)$ and $f(0) = 0$ implies $\widetilde{u}_s f \perp \widetilde{u}_s$. This is a little more complicated than the analogous step in the H-dominant case. Write $f = zw$, with $w = S^* f \in \operatorname{Ran} H_{\widetilde{\psi}_s} \cap \mathcal{H}^\infty$. We have

$$H_u(\widetilde{u}_s f) = H_u(z \widetilde{u}_s w) = K_u(\widetilde{u}_s w) = P(\overline{w} K_u \widetilde{u}_s) = s P(\overline{w} \widetilde{\psi}_s \widetilde{u}_s). \tag{4.1}$$

Since $\overline{w}\widetilde{\psi}_s \in \mathcal{H}^\infty$, we have

$$H_u(\widetilde{u}_s f) = s \widetilde{u}_s \widetilde{\psi}_s \overline{w} = s \widetilde{u}_s \widetilde{\psi}_s z \overline{f} \perp \mathbb{1}. \tag{4.2}$$

Thus,

$$(\widetilde{u}_s f, \widetilde{u}_s) = (\widetilde{u}_s f, u) = (\widetilde{u}_s f, H_u \mathbb{1}) = (\mathbb{1}, H_u(\widetilde{u}_s f)) = 0,$$

as required.

3) Now we can apply Theorem 2.3(ii), which gives $\|f\widetilde{u}_s\| = \|f\|\|\widetilde{u}_s\|$ for all $f \in \operatorname{Ran} H_{\widetilde{\psi}_s}$ and

$$E_K(s) = \widetilde{u}_s \operatorname{Ran} H_{\widetilde{\psi}_s}.$$

Let us check formula (2.7) for the action of K_u on $E_K(s)$; this is a calculation similar to the above:

$$K_u(\widetilde{u}_s f) = P(\overline{f} K_u \widetilde{u}_s) = sP(\overline{f}\widetilde{\psi}_s \widetilde{u}_s) = s\overline{f}\widetilde{\psi}_s\widetilde{u}_s = sH_{\widetilde{\psi}_s} f\widetilde{u}_s,$$

as required.

4) It remains to describe the subspace $E_H(s)$ and the action of H_u on this subspace. By Lemma 2.1, we have $E_H(s) = E_K(s) \cap \widetilde{u}_s^{\perp}$; on the other hand, multiplication by $\widetilde{u}_s/\|\widetilde{u}_s\|$ is an isometry from $\operatorname{Ran} H_{\widetilde{\psi}_s}$ onto $E_K(s)$. Thus, for $f \in \operatorname{Ran} H_{\widetilde{\psi}_s}$, the product $f\widetilde{u}_s$ belongs to $E_H(s)$ if and only if

$$(f\widetilde{u}_s, \widetilde{u}_s) = 0 \quad \Leftrightarrow \quad (f, \mathbb{1}) = 0.$$

Now it is easy to see that

$$\operatorname{Ran} H_{\widetilde{\psi}_s} \cap \mathbb{1}^{\perp} = SS^* \operatorname{Ran} H_{\widetilde{\psi}_s} = S \operatorname{Ran} K_{\widetilde{\psi}_s}.$$

Finally, formula (2.6) for the action of H_u on $E_H(s)$ has already been checked in (4.1)–(4.2). The proof of Theorem 2.2 is now complete.

5. Proof of Theorem 1.5(i) and of Corollary 1.8

5.1. **Frostman shifts and the K-dominant case.** Formulas for the transformation of model spaces under the Frostman shift seem to be folklore among experts; the precise result we need can be found in [6]:

Theorem 5.1. [6, Theorem 10]

(i) *Let θ be an inner function on $\mathbb{D}$, and let $w \in \mathbb{D}$. Let*

$$\alpha_w(z) = \frac{w - z}{1 - \overline{w}z}, \quad g_w(z) = \frac{\sqrt{1 - |w|^2}}{1 - \overline{w}z}.$$

Then $g_w \circ \theta$ is an isometric multiplier from $\mathcal{K}_\theta$ onto $\mathcal{K}_{\alpha_w \circ \theta}$, i.e.

$$(g_w \circ \theta)\mathcal{K}_\theta = \mathcal{K}_{\alpha_w \circ \theta}.$$

(ii) *Let θ_1, θ_2 be non-constant inner functions on $\mathbb{D}$. If there exists an isometric multiplier from $\mathcal{K}_{\theta_1}$ onto $\mathcal{K}_{\theta_2}$, then $\theta_1 = \alpha \circ \theta_2$ for some disk automorphism α. If p_1 and p_2 are two such multipliers, then $p_2 = \gamma p_1$ for a unimodular constant γ.*

In particular, if both θ_1 and θ_2 in Theorem 5.1(ii) vanish at the origin, then $\theta_1 = \gamma' \theta_2$ for a unimodular constant γ'. It follows that the parameters p and θ in the weighted model space $p \operatorname{Ran} H_\theta$ are uniquely defined up to unimodular multiplicative constants, as claimed in Remark 1.4 above.

Proof of Theorem 1.5(i). As already explained, if s is H-dominant, then the first part of Theorem 2.2 immediately provides the required decomposition (1.11) with $p = \mathbb{1}_s/\|\mathbb{1}_s\|$ and $\theta = \psi_s$. Uniqueness of ψ_s is an easy consequence of Remark 1.4. Indeed, by this Remark, if ψ'_s is another inner function satisfying (2.2), then $\psi'_s = e^{i\alpha}\psi_s$ for some $\alpha \in \mathbb{R}$. By (2.3), we have

$$T_{\mathbb{1}_s} H_{\psi_s} = T_{\mathbb{1}_s} H_{\psi'_s} \text{ on } \operatorname{Ran} H_{\psi_s},$$

which implies $\psi_s = \psi'_s$.

Let us consider the K-dominant case. We first observe that for an inner function ψ,

$$\operatorname{Ran} K_\psi = S^* \operatorname{Ran} H_\psi = S^* \mathcal{K}_{z\psi} = \mathcal{K}_\psi.$$

Next, taking $w = \psi(0)$ in Theorem 5.1(i), we obtain

$$\mathcal{K}_\psi = \frac{1}{g_w \circ \psi} \mathcal{K}_{\alpha_w \circ \psi} = \frac{1 - \overline{\psi(0)}\psi}{\sqrt{1 - |\psi(0)|^2}} \operatorname{Ran} H_{S^*(\alpha_w \circ \psi)},$$

with the multiplier $1/(g_w \circ \psi)$ acting isometrically on the model space in the right hand side. Applying this to $\psi = \widetilde{\psi}_s$, by Theorem 2.2(ii) we arrive at

$$E_H(s) = \widetilde{u}_s S \operatorname{Ran} K_{\widetilde{\psi}_s} = p \operatorname{Ran} H_\theta,$$

$$\theta(z) = \frac{\widetilde{\psi}_s(z) - \widetilde{\psi}_s(0)}{z\big(1 - \overline{\widetilde{\psi}_s(0)}\widetilde{\psi}_s(z)\big)}, \quad p(z) = z\widetilde{u}_s(z)\big(1 - \overline{\widetilde{\psi}_s(0)}\widetilde{\psi}_s(z)\big),$$

where $p/\|p\|$ is an isometric multiplier on $\operatorname{Ran} H_\theta$.

Let us check formula for the action of H_u on $E_H(s)$. By (2.6), we have

$$H_u(\widetilde{u}_s f) = s\widetilde{u}_s S H_{\widetilde{\psi}_s} f, \quad f \in S \operatorname{Ran} K_{\widetilde{\psi}_s}.$$

Write

$$f = z\big(1 - \overline{\widetilde{\psi}_s(0)}\widetilde{\psi}_s\big)h, \quad h \in \operatorname{Ran} H_\theta.$$

Now we have

$$SH_{\widetilde{\psi}_s} f = z\widetilde{\psi}_s \overline{f} = \widetilde{\psi}_s(1 - \widetilde{\psi}_s(0)\overline{\widetilde{\psi}_s})\overline{h} = (\widetilde{\psi}_s - \widetilde{\psi}_s(0))\overline{h} = z(1 - \overline{\widetilde{\psi}_s(0)}\widetilde{\psi}_s)\theta\overline{h}.$$

Thus, we obtain

$$H_u(ph) = H_u(\widetilde{u}_s f) = s\widetilde{u}_s z(1 - \overline{\widetilde{\psi}_s(0)}\psi_s)\theta\overline{h} = sp\theta\overline{h},$$

as required. Uniqueness follows from Remark 1.4 as in the first part of the proof. $\square$

5.2. **The self-adjoint case.**

Proof of Corollary 1.8. Let $s = |\lambda|$; then

$$\operatorname{Ker}(H_u - \lambda I) \subset E_H(s).$$

Clearly, H_u maps $\mathcal{H}^2_{\mathrm{real}}$ to $\mathcal{H}^2_{\mathrm{real}}$, and therefore the spectral projections of H_u^2 satisfy the same property. In particular, the orthogonal projection P_s onto $E_H(s)$ satisfies this property. By Theorem 1.5, we have

$$E_H(s) = p \operatorname{Ran} H_\theta$$

for some θ and p. We recall that by Remark 1.4, the functions θ and p are uniquely defined up to unimodular multiplicative constants. Let us check that this constant in the definition of p can be chosen such that $p \in \mathcal{H}^2_{\mathrm{real}}$.

Let m be the smallest integer such that $p^{(m)}(0) \neq 0$; then we have $p(z) = z^m w(z)$ with $w \in \mathcal{H}^2$ and $w(0) \neq 0$. Then for any $f \in \operatorname{Ran} H_\theta$, setting $g = pf$, we get

$$(g, \overline{w(0)}p) = w(0)(pf, p) = w(0)(f, \mathbb{1}) = w(0)f(0) = (z^m w f, z^m) = (g, z^m),$$

which shows that $\overline{w(0)}p = P_s z^m$. Thus, we can choose

$$p = \frac{P_s z^m}{\|P_s z^m\|}$$

which gives $p \in \mathcal{H}^2_{\mathrm{real}}$.

Now let us choose the unimodular constant in the definition of θ such that

$$H_u(p) = \lambda p \theta;$$

then $\theta \in \mathcal{H}^2_{\mathrm{real}}$ and on $\operatorname{Ran} H_\theta$ we have

$$H_u T_p = \lambda T_p H_\theta.$$

The elements of $\operatorname{Ker}(H_u - \lambda I) \cap \mathcal{H}^2_{\mathrm{real}}$ are exactly the elements of the form pf, where $f \in \operatorname{Ran} H_\theta \cap \mathcal{H}^2_{\mathrm{real}}$ satisfies

$$H_u T_p f = \lambda T_p f,$$

which is equivalent to the condition $H_\theta f = f$. $\square$

6. An Example

Here we take a break from our main line of exposition to discuss a simple special case when H_u and K_u have only one or two singular values. In this case, it is easy to describe all relevant subspaces in an elementary way. In a companion paper [11], we consider the case when both H_u and K_u have finitely many singular values and give explicit formulas for the symbol in terms of the singular values and the inner functions ψ_s, $\widetilde{\psi}_s$ appearing in Theorem 2.2.

Theorem 6.1. *Let s, $\widetilde{s}$ be positive numbers and ψ, $\widetilde{\psi}$ be inner functions. The following conditions are equivalent for $u \in \mathrm{BMOA}(\mathbb{T})$:*

(i) *The pair* (H_u, K_u) *has precisely two positive singular values* $s, \widetilde{s}$, *with* s *being* H*-dominant and* $\widetilde{s}$ *being* K*-dominant, and* $\psi = \psi_s$ *and* $\widetilde{\psi} = \widetilde{\psi}_{\widetilde{s}}$ *are the parameters of Theorem 2.2.*

(ii) *We have* $s > \widetilde{s} > 0$ *and*

$$u(z) = \frac{(s^2 - \widetilde{s}^{\,2})\psi(z)}{s - \widetilde{s}z\psi(z)\widetilde{\psi}(z)}, \quad z \in \mathbb{D}.$$

Proof. Assume (i). Since s is the only H-dominant singular value, the spectral decomposition of H_u^2 implies $u = u_s \in E_H(s)$, hence $u = s\psi\mathbb{1}_s$. Multiplying $H_u^2\mathbb{1}_s = s^2\mathbb{1}_s$ by ψ, we obtain

$$\psi H_u u = su. \tag{6.1}$$

Applying the identity $K_u^2 h = H_u^2 h - (h, u)u$ to $h = u$, we infer

$$K_u^2 u = (s^2 - \|u\|^2)u.$$

It follows that $u = \widetilde{u}_{\widetilde{s}}$ and

$$s^2 - \|u\|^2 = \widetilde{s}^{\,2}. \tag{6.2}$$

This gives $s > \widetilde{s}$. Further, by our assumption, it follows that

$$K_u u = \widetilde{s}\widetilde{\psi}u. \tag{6.3}$$

Applying $SS^*f = f - (f, \mathbb{1})\mathbb{1}$ to $f = H_u u$, we obtain

$$SK_u u = H_u u - \|u\|^2\mathbb{1}.$$

Multiplying this identity by ψ and applying (6.1), (6.2), (6.3), we conclude

$$\widetilde{s}z\psi(z)\widetilde{\psi}(z)u(z) = su(z) - (s^2 - \widetilde{s}^{\,2})\psi(z).$$

This gives the formula for u in (ii).

Assume (ii). Denote

$$h(z) = \frac{1}{s - \widetilde{s}z\psi(z)\widetilde{\psi}(z)}.$$

Let us compute $H_u h$. Performing the computations on the unit circle $|z| = 1$, we have

$$\begin{aligned} H_u h &= (s^2 - \widetilde{s}^{\,2})P\left(\frac{\psi}{(s - \widetilde{s}z\psi\widetilde{\psi})(s - \widetilde{s}\overline{z}\overline{\psi}\overline{\widetilde{\psi}})}\right) \\ &= (s^2 - \widetilde{s}^{\,2})P\left(\psi\,\frac{z\psi\widetilde{\psi}}{(s - \widetilde{s}z\psi\widetilde{\psi})(sz\psi\widetilde{\psi} - \widetilde{s})}\right). \end{aligned}$$

Applying the elementary identity

$$\frac{(s^2 - \widetilde{s}^{\,2})\zeta}{(s - \widetilde{s}\zeta)(s\zeta - \widetilde{s})} = \frac{s}{s - \widetilde{s}\zeta} + \frac{\widetilde{s}}{s\zeta - \widetilde{s}}$$

to $\zeta = z\psi(z)\widetilde{\psi}(z)$, we observe that

$$
\begin{aligned}
(s^2 - \widetilde{s}^{\,2})\psi \, \frac{z\psi\widetilde{\psi}}{(s - \widetilde{s}z\psi\widetilde{\psi})(sz\psi\widetilde{\psi} - \widetilde{s})} &= \frac{s\psi}{s - \widetilde{s}z\psi\widetilde{\psi}} + \frac{\widetilde{s}\psi}{sz\psi\widetilde{\psi} - \widetilde{s}} \\
&= \frac{s\psi}{s - \widetilde{s}z\psi\widetilde{\psi}} + \frac{\widetilde{s}\overline{z}\overline{\widetilde{\psi}}}{s - \widetilde{s}\overline{z}\overline{\psi}\overline{\widetilde{\psi}}}.
\end{aligned}
$$

We note that the first term in the right hand side is in $\mathcal{H}^2(\mathbb{T})$ and the second one is orthogonal to $\mathcal{H}^2(\mathbb{T})$. Consequently,

$$
H_u h = \frac{s\psi}{s - \widetilde{s}z\psi\widetilde{\psi}} = s\psi h = \frac{s}{s^2 - \widetilde{s}^{\,2}} u.
$$

Using (2.11), we obtain

$$
H_u^2 h = H_u(s\psi h) = sP(\overline{\psi} H_u h) = sP(\overline{\psi} s\psi h) = s^2 h,
$$

and so $h, u \in E_H(s)$. Therefore s is the only H-dominant singular value with the inner parameter of Theorem 2.2 being $\psi_s = \psi$.

It remains to compute

$$
K_u u = S^* H_u u = (s^2 - \widetilde{s}^{\,2}) s S^* h.
$$

Observe that

$$
sh(z) = \frac{1}{1 - \frac{\widetilde{s}}{s} z\psi\widetilde{\psi}} = 1 + \frac{\widetilde{s}z\psi\widetilde{\psi}}{s - \widetilde{s}z\psi\widetilde{\psi}},
$$

and so

$$
sS^* h = \frac{\widetilde{s}\psi\widetilde{\psi}}{s - \widetilde{s}z\psi\widetilde{\psi}}.
$$

This gives

$$
K_u u = \widetilde{s}\widetilde{\psi} u.
$$

Using (2.11), from here we get

$$
K_u^2 u = \widetilde{s} K_u(\widetilde{\psi} u) = \widetilde{s} P(\overline{\widetilde{\psi}} K_u u) = \widetilde{s}^2 u.
$$

It follows that $\widetilde{s}$ is the only K-dominant singular value with the inner parameter of Theorem 2.2 being $\widetilde{\psi}_{\widetilde{s}} = \widetilde{\psi}$. $\square$

Finally, we justify the claim made in Remark 1.7. Let u be as in the above Theorem. Observe that $\widetilde{u}_{\widetilde{s}} = u$ and the inner-outer factorisation of u is given by

$$
u(z) = \psi(z) u_0(z), \quad u_0(z) = \frac{s^2 - \widetilde{s}^2}{s - \widetilde{s} z \psi(z) \widetilde{\psi}(z)}.
$$

Thus, by Theorem 2.2(ii), we have the weighted model space representation for the Schmidt space

$$
E_{K_u}(s) = E_{H_{S^* u}}(s) = u \operatorname{Ran} H_{\widetilde{\psi}_s} = \psi u_0 \operatorname{Ran} H_{\widetilde{\psi}}.
$$

According to Theorem 6.1, the inner functions ψ, $\widetilde{\psi}$ here can be chosen in an arbitrary way.

7. Proof of Theorem 1.5(ii)

Our arguments in this section follow closely the original AAK paper [1]; however, the last part of the proof is somewhat simpler, avoiding the perturbation argument of [1]. Throughout the section, we use the commutation relation (2.12). We denote by $n(s;H_u)$ (resp. by $n[s;H_u]$) the total multiplicity of the spectrum of the self-adjoint operator $|H_u|$ in the interval (s,∞) (resp. $[s,\infty)$).

7.1. **The case $s = \|H_u\|$.** The first statement concerns the case $s = \|H_u\|$; it contains Theorem 1.5(ii) in the case $n(s;H_u) = 0$.

Lemma 7.1. *Let $u \in \mathrm{BMOA}(\mathbb{T})$, $s = \|H_u\| > 0$ and $E_H(s) \neq \{0\}$. If $f \in E_H(s)$ and a is an inner divisor of f, then $f/a \in E_H(s)$ and*

$$H_u(f/a) = T_a H_u f. \tag{7.1}$$

Furthermore, in the weighted model space representation $E_H(s) = p \operatorname{Ran} H_\theta$, the isometric multiplier p is an outer function.

Proof. First observe that since $s = \|H_u\|$, condition $f \in E_H(s)$ is equivalent to

$$\|H_u(f)\| = s\|f\|.$$

Next, let $f = af_0 \in E_H(s)$, where a is inner and $f_0 \in \mathcal{H}^2$. We have

$$T_{\overline{a}} H_u f_0 = H_u T_a f_0 = H_u f$$

and therefore

$$s\|f_0\| = s\|f\| = \|H_u f\| = \|T_{\overline{a}} H_u f_0\| \leq \|H_u f_0\|;$$

it follows that $s\|f_0\| = \|H_u f_0\|$ and therefore $f_0 \in E_H(s)$. Furthermore, using the fact that $T_a T_{\overline{a}}$ is an orthogonal projection (as T_a is an isometry), from $\|T_a T_{\overline{a}} H_u f_0\| = \|H_u f_0\|$ we conclude

$$T_a T_{\overline{a}} H_u f_0 = H_u f_0.$$

The last identity can be rewritten as

$$T_a H_u f = H_u f_0,$$

which is the same as (7.1). Finally, applying the above statement to $f = p$, we obtain that the outer factor p_0 of p is in $E_H(s)$, and so it can be represented as $p_0 = ph$ with $h \in \operatorname{Ran} H_\theta$. This implies that $p_0 = p$. $\square$

7.2. **Introducing the AAK unimodular symbol.**

Lemma 7.2. *For $u \in \mathrm{BMOA}(\mathbb{T})$ and $s > 0$, let $f \in E_H(s)$, $f \neq 0$. Then the ratio $\phi = H_u f/(s\overline{f})$ is a unimodular function on $\mathbb{T}$ which is independent of the choice of $f \in E_H(s)$.*

Proof. In order to prove that ϕ is independent of the choice of f, take $f_1, f_2 \in E_H(s)$ and let $H_u f_1 = sg_1$, $H_u f_2 = sg_2$; it suffices to check that

$$\overline{f_1} g_2 = \overline{f_2} g_1. \tag{7.2}$$

For $n \geq 0$, we have

$$\begin{aligned} s(g_1\overline{f_2}, z^n) = (H_u f_1, z^n f_2) = ((S^*)^n H_u f_1, f_2) = (H_u S^n f_1, f_2) \\ = (H_u f_2, S^n f_1) = s(g_2\overline{f_1}, z^n). \end{aligned}$$

Similarly, we get

$$(\overline{g_1} f_2, z^n) = (\overline{g_2} f_1, z^n), \quad n \geq 0,$$

and so (7.2) follows. Finally, the fact that $|\phi| = 1$ comes from applying (7.2) to $f_2 = g_1$. $\square$

Let ϕ be as in the previous lemma and set $v = sP(\phi)$.

Lemma 7.3. *Under the above conditions,*

$$n(s; H_u) \leq \operatorname{rank} H_{u-v} \leq \deg \varphi.$$

Proof. For any $f \in E_H(s)$ we have, by the definition of ϕ,

$$H_v f = sP(\phi\overline{f}) = H_u f$$

and so the Hankel operator H_{u-v} vanishes on the subspace $E_H(s)$. In particular, it vanishes on p and consequently it vanishes on the minimal S-invariant subspace containing p. Since the inner factor of p is φ, we conclude that

$$\varphi\mathcal{H}^2 \subset \operatorname{Ker} H_{u-v} \tag{7.3}$$

or equivalently

$$\operatorname{Ran} H_{u-v} \subset \mathcal{K}_\varphi;$$

it follows that

$$\operatorname{rank} H_{u-v} \leq \dim \mathcal{K}_\varphi = \deg \varphi.$$

Further, we have

$$\|H_u - H_{u-v}\| = \|H_v\| \leq s\|\phi\|_{L^\infty} = s.$$

Putting this together, we obtain

$$n(s; H_u) = n(s; H_v + H_{u-v}) \leq n(s; H_v) + \operatorname{rank} H_{u-v} \leq 0 + \deg \varphi = \deg \varphi,$$

as required. $\square$

7.3. **Completing the proof.** Denote $w = T_{\overline{\varphi}}u$ and consider the Hankel operator

$$H_w = H_u T_\varphi = T_{\overline{\varphi}} H_u.$$

Observe that by (7.3) we have $H_w = H_v T_\varphi$ and so $\|H_w\| \leq \|H_v\| \leq s$; in fact, by the following lemma we have $\|H_w\| = s$.

Lemma 7.4. *We have*

$$ap_0 \operatorname{Ran} H_\theta \subset E_{H_w}(s)$$

for any $a|\varphi$, i.e. for any inner divisor a of φ. In particular,

$$E_{H_u}(s) = p \operatorname{Ran} H_\theta \subset E_{H_w}(s).$$

Proof. For every $f \in \operatorname{Ran} H_\theta$ we have

$$T_{\overline{\varphi}} H_u(pf) = sT_{\overline{\varphi}}(pH_\theta f) = sT_{\overline{\varphi}}(\varphi p_0 H_\theta f) = sp_0 H_\theta(f),$$
$$T_{\overline{\varphi}} H_u(p_0 H_\theta f) = H_u(\varphi p_0 H_\theta f) = H_u(pH_\theta f) = spH_\theta^2 f = spf,$$

and so $p \operatorname{Ran} H_\theta \subset E_{H_w}(s)$. Thus, we can apply Lemma 7.1 (with H_w in place of H_u), which yields that $ap_0 f \in E_{H_w}(s)$ for any $a|\varphi$, as required. □

At the last two steps of the proof below, we depart from the original argument of AAK. The AAK proof involves dimension count if $\deg\theta < \infty$ and a (rather tricky) approximation argument if $\deg\theta = \infty$. Instead, we proceed by considering a quotient space, which works for all values of $\deg\theta$.

The following lemma is purely operator theoretic and does not use any specifics of Hankel operators.

Lemma 7.5. *With the above notation, we have*

$$\dim E_{H_w}(s) \ominus E_{H_u}(s) \leq n(s; H_u).$$

Proof. On the space $E = E_{H_w}(s)$ we consider the self-adjoint operator

$$A = P_E H_u^2|_E,$$

where P_E is the orthogonal projection onto E. First note that by a standard variational argument,

$$n(s^2; A) \leq n(s^2; H_u^2).$$

This follows, for example, by writing the minimax principle in the form [4, Theorem 10.2.3]

$$n(s^2; A) = \sup\{\dim L : L \subset E,\ (Af, f) > s^2\|f\|^2 \quad \forall f \in L \setminus \{0\}\},$$

and comparing with a similar expression for $n(s^2; H_u^2)$.

Next, we notice that

$$H_w^2 = H_u T_\varphi T_{\overline{\varphi}} H_u \leq H_u^2$$

and therefore $A \geq s^2 I$. Since $E_{H_u}(s) \subset E$, it is straightforward to see that

$$E_{H_u}(s) = \operatorname{Ker}(A - s^2 I).$$

Putting this together, we obtain

$$\dim E \ominus E_{H_u}(s) = n(s^2; A) \le n(s^2; H_u^2) = n(s; H_u),$$

as required. $\square$

Our final step is

Lemma 7.6. *With the above notation, we have*

$$\deg \varphi \le \dim E_{H_w}(s) \ominus E_{H_u}(s).$$

Proof. Since (by Lemma 7.4)

$$p_0 \operatorname{span}\{a : a|\varphi\} \operatorname{Ran} H_\theta \subset E_{H_w}(s)$$

and $E_{H_u}(s) = p_0\varphi \operatorname{Ran} H_\theta$, we get, replacing the orthogonal complement by the algebraic quotient space,

$$\begin{aligned}\dim E_{H_w}(s) \ominus E_{H_u}(s) &= \dim E_{H_w}(s)/E_{H_u}(s)\\ &\ge \dim(p_0 \operatorname{span}\{a : a|\varphi\} \operatorname{Ran} H_\theta / p_0\varphi \operatorname{Ran} H_\theta).\end{aligned}$$

Since multiplication by p_0 is an injective operation, we have

$$\begin{aligned}&\dim(p_0 \operatorname{span}\{a : a|\varphi\} \operatorname{Ran} H_\theta / p_0\varphi \operatorname{Ran} H_\theta)\\ &\qquad = \dim(\operatorname{span}\{a : a|\varphi\} \operatorname{Ran} H_\theta / \varphi \operatorname{Ran} H_\theta)\\ &\qquad = \dim(\operatorname{span}\{a : a|\varphi\} \operatorname{Ran} H_\theta \ominus \varphi \operatorname{Ran} H_\theta)\\ &\qquad \ge \dim \operatorname{span}\{a - (a,\varphi)\varphi : a|\varphi\},\end{aligned}$$

because for $a|\varphi$ and $f \in \mathcal{H}^2(\mathbb{T})$

$$\begin{aligned}(a - (a,\varphi)\varphi, \varphi f) = (a, \varphi f) - (a,\varphi)(\mathbb{1}, f) &= (\mathbb{1}, \overline{a}\varphi f) - (a,\varphi)(\mathbb{1}, f)\\ &= (\mathbb{1}, \overline{a}\varphi)(\mathbb{1}, f) - (a,\varphi)(\mathbb{1}, f) = 0.\end{aligned}$$

Finally, it is elementary to observe that

$$\dim \operatorname{span}\{a - (a,\varphi)\varphi : a|\varphi\} = \deg \varphi.$$

$\square$

Proof of Theorem 1.5(ii). Putting together the last two lemmas, we obtain $\deg \varphi \le n(s; H_u)$. Combining this with Lemma 7.3, we arrive at the desired conclusion

$$\deg \varphi = n(s; H_u) = \operatorname{rank} H_{u-v}.$$

$\square$

8. Proof of Theorem 1.9

We consider the conformal map

$$\mu : \mathbb{C}_+ \to \mathbb{D}, \quad \mu(z) = \frac{z-i}{z+i},$$

and the corresponding unitary operator $U_\mu : \mathcal{H}^2(\mathbb{T}) \to \mathcal{H}^2(\mathbb{R})$,

$$U_\mu f = \mathbf{f}, \quad \mathbf{f}(z) = \frac{1}{\sqrt{\pi}(z+i)} f(\mu(z)), \quad \operatorname{Im} z > 0. \tag{8.1}$$

It is evident that U_μ maps a Beurling subspace $\theta\mathcal{H}^2(\mathbb{T})$ onto $(\theta \circ \mu)\mathcal{H}^2(\mathbb{R})$, and therefore

$$U_\mu \mathcal{K}_\theta = \boldsymbol{\mathcal{K}}_{\theta\circ\mu}.$$

From here one reads off the formula for the action of U_μ on weighted model spaces: if p is an isometric multiplier on $\mathcal{K}_\theta$, then

$$U_\mu(p\mathcal{K}_\theta) = \mathbf{p}\boldsymbol{\mathcal{K}}_{\theta\circ\mu}, \quad \mathbf{p} = p \circ \mu, \tag{8.2}$$

and $\mathbf{p}$ is an isometric multiplier on $\boldsymbol{\mathcal{K}}_{\theta\circ\mu}$. Next, we have a straightforward

Proposition 8.1. *Let* $\mathbf{u} \in \mathrm{BMOA}(\mathbb{R})$ *and let* U_μ *be as in* (8.1). *Then we have*

$$U_\mu^* \mathbf{H_u} U_\mu = H_{S^*u}, \quad \mathbf{u}(x) = u(\mu(x)).$$

Proof. The following calculation is valid for the dense set of functions $h_1, h_2 \in \mathcal{H}^\infty(\mathbb{T})$ and then the result extends to all $h_1, h_2 \in \mathcal{H}^2(\mathbb{T})$:

$$\begin{aligned}
&(\mathbf{H_u} U_\mu h_1, U_\mu h_2)_{L^2(\mathbb{R})} = (\mathbf{u}, (U_\mu h_1)(U_\mu h_2))_{L^2(\mathbb{R})} \\
&= \int_{-\infty}^{\infty} \mathbf{u}(x)\overline{h_1(\mu(x))h_2(\mu(x))}\frac{dx}{\pi(x-i)^2} = \int_{-\infty}^{\infty} \frac{x+i}{x-i}\mathbf{u}(x)\overline{h_1(\mu(x))h_2(\mu(x))}\frac{dx}{\pi(x^2+1)} \\
&\qquad = \int_{-\pi}^{\pi} e^{-i\theta}u(e^{i\theta})\overline{h_1(e^{i\theta})h_2(e^{i\theta})}\frac{d\theta}{2\pi} = (H_{S^*u}h_1, h_2)_{L^2(\mathbb{T})},
\end{aligned}$$

as required. □

Proof of Theorem 1.9. Let us apply Proposition 8.1; we obtain $U_\mu^* \mathbf{H_u} U_\mu = H_{S^*u} = K_u$, with $u \circ \mu = \mathbf{u}$. By Theorem 1.5(i), we have

$$E_{K_u}(s) = p \operatorname{Ran} H_\theta$$

and

$$K_u T_p = s T_p H_\theta \text{ on } \operatorname{Ran} H_\theta, \tag{8.3}$$

for some inner θ and for an isometric multiplier p on $\operatorname{Ran} H_\theta$. Now let us apply (8.2); this gives

$$\operatorname{Ker}(\mathbf{H_u}^2 - s^2 I) = U_\mu(p \operatorname{Ran} H_\theta) = U_\mu(p\mathcal{K}_{z\theta}) = \mathbf{p}\boldsymbol{\mathcal{K}_\theta} = \mathbf{p} \operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}},$$

with

$$\mathbf{p} = p \circ \mu, \quad \boldsymbol{\theta}(z) = \mu(z)\theta(\mu(z)).$$

Applying U_μ to (8.3), we obtain

$$\mathbf{H_u} U_\mu T_p = s U_\mu T_p H_\theta \text{ on } \operatorname{Ran} H_\theta.$$

Observing that $U_\mu T_p = \mathbf{T_p} U_\mu$, we arrive at

$$\mathbf{H_u} \mathbf{T_p} U_\mu = s \mathbf{T_p} U_\mu H_\theta \text{ on } \operatorname{Ran} H_\theta.$$

By Proposition 8.1, we have $U_\mu H_\theta = U_\mu K_{S\theta} = \mathbf{H}_{\boldsymbol{\theta}} U_\mu$, and so

$$\mathbf{H_u} \mathbf{T_p} U_\mu = s \mathbf{T_p} \mathbf{H}_{\boldsymbol{\theta}} U_\mu \text{ on } \operatorname{Ran} H_\theta.$$

Finally, by (8.2) we have $U_\mu \operatorname{Ran} H_\theta = U_\mu \mathcal{K}_{S\theta} = \operatorname{Ran} \mathbf{H}_{\boldsymbol{\theta}}$. This completes the proof of part (i).

Part (ii) is a direct consequence of Theorem 1.5(ii) and the fact that the degree of the inner factor of p is invariant under the composition with μ. □

Appendix A. Linear Hankel operators

Here we rewrite Theorem 1.5(i) in terms of a representation for the Hankel matrix Γ as a linear (rather than anti-linear) operator on the Hardy space. Let J be the linear involution in $L^2(\mathbb{T})$,

$$Jf(z) = f(\overline{z}), \quad z \in \mathbb{T},$$

and let $\mathbf{C}$ be the anti-linear involution in $\mathcal{H}^2(\mathbb{T})$,

$$\mathbf{C}f(z) = \overline{f(\overline{z})}, \quad z \in \mathbb{T}.$$

For a symbol $u \in \operatorname{BMOA}(\mathbb{T})$, let us define the *linear* Hankel operator G_u in $\mathcal{H}^2(\mathbb{T})$ by

$$G_u f = P(u \cdot Jf).$$

It is straightforward to see that

$$(G_u z^n, z^m) = \widehat{u}(n+m),$$

and so G_u is unitarily equivalent to the Hankel matrix $\Gamma = \{\widehat{u}(n+m)\}_{n,m=0}^\infty$.

Theorem A.1. *Let s be a singular value of G_u. Then there exists an inner function θ and an isometric multiplier p on $\operatorname{Ran} H_\theta$ such that*

$$\operatorname{Ker}(G_u^* G_u - s^2 I) = \mathbf{C}(p \operatorname{Ran} H_\theta).$$
$$\operatorname{Ker}(G_u G_u^* - s^2 I) = p \operatorname{Ran} H_\theta.$$

The action

$$G_u : \operatorname{Ker}(G_u^* G_u - s^2 I) \to \operatorname{Ker}(G_u G_u^* - s^2 I)$$

is given by

$$G_u \mathbf{C}(pf) = s p \theta \overline{f}, \quad f \in \operatorname{Ran} H_\theta.$$

This theorem immediately follows from Theorem 1.5 after identification

$$G_u = H_u \mathbf{C}, \quad G_u^* = \mathbf{C} H_u.$$

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